**Porous catalyst:** Consider a reaction occurring inside a catalyst pore. Normally the surface area of the outer surface of the catalyst particle is negligible compared to the surface area available in the pores.

Let the pore be a cylinder of radius $r$ and length $L$. The gas diffuses into the pore and reacts on the pore walls. The reaction and diffusion rates are comparable, i.e. we don’t assume that one is much faster than the other. Then *progressive conversion* model is more appropriate, *shrinking core model* is not.

In this case, the reaction occurs on the *side surface* of the cylinder. In comparison to that area, the outer surface area of the particle and the area at the end of the cylinder can be neglected. We assume that there is no radial variation in the species concentration. The mass balance for this section gives

$$\pi r^2 \mathcal{D} \left( -\frac{\partial C_A}{\partial x} \right)_x = \pi r^2 \mathcal{D} \left( -\frac{\partial C_A}{\partial x} \right)_{x+\Delta x} + k_s C_A (2\pi r \Delta x)$$

This can be rearranged as

$$\mathcal{D} \left( \frac{\partial C_A}{\partial x} \right)_{x+\Delta x} - \left( \frac{\partial C_A}{\partial x} \right)_x = \frac{2k_s C_A}{r} \Delta x$$

At the limit, we get

$$\left( \frac{\partial^2 C_A}{\partial x^2} \right) - \frac{2k_s C_A}{\mathcal{D} r} = 0$$

The reaction rate constant based on surface ($k_s$) and based on volume ($k$) can be related as follows. (Naturally this depends on the surface area per unit mass of catalyst and the mass of catalyst per unit volume of the reactor).
\[ k \times \pi r^2 L = k_r \times 2\pi r L \]

Therefore, \( k = \frac{2k}{r} \), which leads to \( \left( \frac{\partial^2 C_A}{\partial x^2} \right) - \frac{k C_A}{D} = 0 \)

The solution of the ODE is \( C_A = M_1 e^{mx} + M_2 e^{-mx} \) where \( m = \frac{k}{\sqrt{D}} \). The boundary conditions are \( (C_A)_{x=0} = C_{As} \) and \( \left( \frac{dC_A}{dx} \right)_{x=L} = 0 \) After substitution and rearrangement we can show that the following expression holds true.

\[
\frac{C_A}{C_{As}} = \frac{\cosh \left( m(L - x) \right)}{\cosh mL}
\]

The effectiveness factor \( E \) is defined as the actual reaction rate within the pore divided by the rate if not slowed by diffusion in the pore. This is the same as \( \frac{(C_A)_{AV}}{C_{As}} \), which means we have to calculate \( \frac{(C_A)_{AV}}{C_{As}} = \int_{x=0}^{x=L} \frac{\cosh \left( m(L - x) \right)}{\cosh mL} \, dx / L \).

\[
\frac{1}{L} \int_{x=0}^{x=L} \frac{\cosh \left( m(L - x) \right)}{\cosh mL} \, dx = \frac{1}{L} \int_{x=0}^{x=L} \cosh \left( m(L - x) \right) \, dx
\]

\[
= \frac{1}{mL} \left( \frac{\sinh \left( m(L - x) \right)}{\cosh mL} \right)_{x=0}^{x=L} = \frac{1}{mL} \left( \frac{\sinh \left( mL \right)}{\cosh mL} - \frac{\sinh \left( 0 - L \right)}{\cosh mL} \right)
\]

\[
= \frac{1}{mL} \left( \frac{\sinh \left( mL \right)}{\cosh mL} \right) = \frac{\tanh \left( mL \right)}{mL}
\]

Therefore, (for the first order reaction) effectiveness factor \( E_{\text{first order}} = \frac{\tanh \left( mL \right)}{mL} \)

For other kinetics, similar expressions can be developed.